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Constraint and super Yang-Mills equations on the deformed superspace $\mathbb{R}_\hbar^{(4|16)}$

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Abstract: It has been known for quite some time that the $\mathcal{N} = 4$ super Yang-Mills equations defined on four-dimensional euclidean space are equivalent to certain constraint equations on the euclidean superspace $\mathbb{R}^{(4|16)}$. In this paper we consider the constraint equations on a deformed superspace $\mathbb{R}_\hbar^{(4|16)}$ à la Seiberg and derive the deformed super Yang-Mills equations. In showing this, we propose a super Seiberg-Witten map.

Keywords: Extended Supersymmetry, Superspaces, Non-Commutative Geometry, Integrable Field Theories.

Contents

1. Introduction

In the last couple of years, field theories defined on noncommutative spacetimes have been explored extensively, mainly due to their realization in string theory. In particular, theories on spacetimes endowed with Moyal type deformations have been discussed. Besides purely bosonic deformations, also deformed superspaces became of interest (see, e.g., references [1]–[5] and more recent ones [6, 7, 8]). As it was shown in [9]–[13], deformed superspaces arise quite naturally in string theory, as well. For that reason, it is important to study such spaces as well as theories defined on them. Within the past few months, several authors have dealt with, for instance, deformed versions of the Wess-Zumino model and super Yang-Mills theory. Also quantum aspects [14]–[20] and nonperturbative solutions, such as instantons [21]–[24] have been explored.

However, only deformed $\mathcal{N} \leq 2$ super Yang-Mills theories have been discussed in the literature. This is probably due to the lack of a proper superspace formulation of the actions for $\mathcal{N}=3,4$ super Yang-Mills theory — even in the undeformed case. A loophole to this obstruction is to consider the constraint equations instead of an action. In the

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undeformed setup, it was pointed out in [25] and proven in [26, 27] that there is a one-to-one correspondence between the equations of motion and the aforementioned constraint equations. The latter ones are defined on the superspace $\mathbb{R}^{(4|4\mathcal{N})}$ and amount to a flatness condition on the superconnection.[1]

In this paper we are going to use this fact to derive the equations of motion of deformed super Yang-Mills theory by starting from properly deformed constraint equations. Since $\mathcal{N}=3$ and $\mathcal{N}=4$ super Yang-Mills theory are basically equivalent, we may just consider the $\mathcal{N}=4$ case. In deriving the superfield expansions, we propose a generalization of the Seiberg-Witten map [28] to superspace. For simplicity we shall restrict ourselves to first order in the deformation.

The paper is organized as follows. In section 2, we begin with a brief review on $\mathcal{N}=4$ super Yang-Mills theory which is defined on four-dimensional euclidean spacetime. In section 3, we then introduce the deformed superspace $\mathbb{R}_\hbar^{(4|4\mathcal{N})}$. Having fixed the setup, we start in section 4 from the deformed constraint equations and derive the deformed equations of motion by using the abovementioned Seiberg-Witten map. Finally, in the appendix A we briefly review the expansion of the undeformed superfields.

2. $\mathcal{N}=4$ super Yang-Mills theory

2.1 Generalities

We begin our considerations by fixing our notation and conventions, which to large extent coincide with those of [29]. First of all, we shall always make the identification

$$x^\mu \sim x^{\alpha\dot\alpha} \equiv -\frac{i}{2}\bar\sigma_\mu^{\alpha\dot\alpha}x^\mu\,, \tag{2.1}$$

where $(\bar\sigma_\mu^{\alpha\dot\alpha}) = (1, -i\vec\sigma)$ and $\alpha,\beta,\ldots,\dot\alpha,\dot\beta,\ldots = 1,2$. Isospin indices will be denoted by small Latin letters starting from the middle of the alphabet, i.e., $i,j,\ldots = 1,\ldots,4$. Moreover, we use

$$(\epsilon^{\alpha\beta}) = (\epsilon^{\dot\alpha\dot\beta}) = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix} \qquad \text{and} \qquad (\epsilon_{\alpha\beta}) = (\epsilon_{\dot\alpha\dot\beta}) = \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}, \tag{2.2}$$

with $\epsilon_{\alpha\delta}\epsilon^{\delta\beta} = \delta_\alpha^\beta$ and $\epsilon_{\dot\alpha\dot\delta}\epsilon^{\dot\delta\dot\beta} = \delta_{\dot\alpha}^{\dot\beta}$. Spinors with upper and lower indices are related via the ϵ-tensors, i.e.,

$$\psi^{i\alpha} = \epsilon^{\alpha\beta}\psi^i_\beta \qquad \text{and} \qquad \psi^i_\alpha = \epsilon_{\alpha\beta}\psi^{i\beta}\,. \tag{2.3}$$

Similarly, we have for the dotted ones

$$\bar\psi_i^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\bar\psi_{i\dot\beta} \qquad \text{and} \qquad \bar\psi_{i\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\bar\psi_i^{\dot\beta}\,. \tag{2.4}$$

It is important to stress that on euclidean space there is no relation between undotted and dotted spinors. Throughout this paper we use the following spinor summation convention:

$$\begin{aligned}\psi\chi &= \psi^{i\alpha}\chi^i_\alpha = -\psi^i_\alpha\chi^{i\alpha} = \chi^{i\alpha}\psi^i_\alpha = \chi\psi\,,\\ \bar\psi\bar\chi &= \bar\psi_{i\dot\alpha}\bar\chi_i^{\dot\alpha} = -\bar\psi_i^{\dot\alpha}\bar\chi_{i\dot\alpha} = \bar\chi_{i\dot\alpha}\bar\psi_i^{\dot\alpha} = \bar\chi\bar\psi\,.\end{aligned} \tag{2.5}$$

[1] Note that even after a successful deformation of the equations of motion, one still had to find the corresponding action (in component fields) for a full description.

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2.2 Euclidean $\mathcal{N}=4$ super Yang-Mills action

To write down the super Yang-Mills action, we recall first that the automorphism group of $\mathcal{N}=4$ supersymmetry on four-dimensional euclidean space is SO(5,1). The $\mathcal{N}=4$ supermultiplet consists of a gauge field $A_{\alpha\dot{\alpha}}$, six scalars $W_{ij}=-W_{ji}$ and eight Weyl fermions χ^i_α and $\bar{\chi}_{i\dot{\alpha}}$. All of these fields are subject to a specific reality condition induced by the anti-linear involutive automorphism σ:[2,3]

$$
\begin{aligned}
\sigma(A_{\alpha\dot{\beta}}) &= -\epsilon_{\alpha\beta}\epsilon_{\dot{\beta}\dot{\gamma}}(A_{\beta\dot{\gamma}})^\dagger\,, && (2.6a)\\
\sigma(W_{ij}) &= -T_i^k(W_{kl})^\dagger T_j^l\,, && (2.6b)\\
\sigma(\chi^i_\alpha) &= \epsilon_{\alpha\beta}T^i_j(\chi^j_\beta)^\dagger\,, && (2.6c)\\
\sigma(\bar{\chi}_{i\dot{\alpha}}) &= \epsilon_{\dot{\alpha}\dot{\beta}}T_i^j(\bar{\chi}_{j\dot{\beta}})^\dagger\,. && (2.6d)
\end{aligned}
$$

The matrix (T^i_j) is given by the following expression:

$$
(T^i_j)=\begin{pmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\end{pmatrix}. \qquad (2.6e)
$$

For a detailed review on supersymmetry, especially on euclidean spaces we refer the reader to reference [30]. Note that all of the above fields live in the adjoint representation of some compact gauge group G.

Using the conventions given in the previous subsection, the $\mathcal{N}=4$ super Yang-Mills action on $\mathbb{R}^4$ takes the following form

$$
\begin{aligned}
S = \int d^4x\,\mathrm{tr}\Big\{ &-\frac{1}{2}\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}(\nabla_{\alpha\dot{\alpha}}W^{ij})(\nabla_{\beta\dot{\beta}}W_{ij})+\epsilon^{\dot{\alpha}\dot{\gamma}}\epsilon^{\dot{\beta}\dot{\delta}}f_{\dot{\alpha}\dot{\beta}}f_{\dot{\gamma}\dot{\delta}}+\epsilon^{\alpha\gamma}\epsilon^{\beta\delta}f_{\alpha\beta}f_{\gamma\delta}-\\
&-\frac{1}{2}\epsilon_{ijkl}\epsilon^{\alpha\beta}\chi^k_\alpha[\chi^l_\beta,W^{ij}]-\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\chi}_{i\dot{\alpha}}[\bar{\chi}_{j\dot{\beta}},W^{ij}]+\\
&+\frac{1}{8}[W^{ij},W^{kl}][W_{ij},W_{kl}]+\epsilon^{\alpha\beta}\epsilon^{\dot{\beta}\dot{\gamma}}\left(\chi^i_\alpha(\nabla_{\beta\dot{\beta}}\bar{\chi}_{i\dot{\gamma}})-(\nabla_{\beta\dot{\beta}}\chi^i_\alpha)\bar{\chi}_{i\dot{\gamma}}\right)\Big\}, \qquad (2.7)
\end{aligned}
$$

where we have abbreviated

$$
W^{ij}\equiv\frac{1}{2}\epsilon^{ijkl}W_{kl}\,.
$$

Moreover, the bosonic curvature is decomposed (in self-dual and anti-self-dual parts) as

$$
[\nabla_{\alpha\dot{\alpha}},\nabla_{\beta\dot{\beta}}]=\epsilon_{\dot{\alpha}\dot{\beta}}f_{\alpha\beta}+\epsilon_{\alpha\beta}f_{\dot{\alpha}\dot{\beta}}\,. \qquad (2.8)
$$

The equations of motion induced by the action (2.7) read as

$$
\begin{aligned}
\epsilon^{\alpha\beta}\nabla_{\alpha\dot{\alpha}}\chi^i_\beta+\frac{1}{2}\epsilon^{ijkl}[W_{kl},\bar{\chi}_{j\dot{\alpha}}] &= 0\,, && (2.9a)\\
\epsilon^{\dot{\alpha}\dot{\beta}}\nabla_{\alpha\dot{\alpha}}\bar{\chi}_{i\dot{\beta}}+[W_{ij},\chi^j_\alpha] &= 0 && (2.9b)
\end{aligned}
$$

[2]Summation over repeated indices is implied.

[3]Recall that a field f is said to be real if it is a fixed point of the involution σ, i.e., $\sigma(f)=f$.

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and

$$\epsilon^{\dot\alpha\dot\beta}\nabla_{\gamma\dot\alpha}f_{\dot\beta\dot\gamma}+\epsilon^{\alpha\beta}\nabla_{\alpha\dot\gamma}f_{\beta\gamma} = \frac{1}{4}\epsilon^{ijkl}[\nabla_{\gamma\dot\gamma}W_{ij},W_{kl}]+\{\chi^i_\gamma,\bar\chi_{i\dot\gamma}\}\,, \quad (2.10a)$$

$$\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\nabla_{\alpha\dot\alpha}\nabla_{\beta\dot\beta}W_{ij}-\frac{1}{4}\epsilon^{klmn}[W_{mn},[W_{kl},W_{ij}]] = \frac{1}{2}\epsilon_{ijkl}\epsilon^{\alpha\beta}\{\chi^k_\alpha,\chi^l_\beta\}+\epsilon^{\dot\alpha\dot\beta}\{\bar\chi_{i\dot\alpha},\bar\chi_{j\dot\beta}\}. \quad (2.10b)$$

The action is invariant under the following supersymmetry transformations

$$\delta_{\xi,\bar\xi}A_{\alpha\dot\alpha} = -\epsilon_{\alpha\beta}\xi^{i\beta}\bar\chi_{i\dot\alpha}+\epsilon_{\dot\alpha\dot\beta}\bar\xi^{\dot\beta}_i\chi^i_\alpha\,, \quad (2.11a)$$

$$\delta_{\xi,\bar\xi}W_{ij} = \epsilon_{ijkl}\xi^{k\alpha}\chi^l_\alpha-\bar\xi^{\dot\alpha}_i\bar\chi_{j\dot\alpha}+\bar\xi^{\dot\alpha}_j\bar\chi_{i\dot\alpha}\,, \quad (2.11b)$$

$$\delta_{\xi,\bar\xi}\chi^i_\alpha = -2\xi^{i\beta}f_{\alpha\beta}+\frac{1}{2}\epsilon_{\alpha\beta}\epsilon^{iklm}\xi^{j\beta}[W_{lm},W_{jk}]-\epsilon^{ijkl}\bar\xi^{\dot\alpha}_j\nabla_{\alpha\dot\alpha}W_{kl}\,, \quad (2.11c)$$

$$\delta_{\xi,\bar\xi}\bar\chi_{i\dot\alpha} = 2\xi^{j\alpha}\nabla_{\alpha\dot\alpha}W_{ij}+2\bar\xi^{\dot\beta}_i f_{\dot\alpha\dot\beta}+\frac{1}{2}\epsilon_{\dot\alpha\dot\beta}\epsilon^{jklm}\bar\xi^{\dot\beta}_j[W_{lm},W_{ik}]\,, \quad (2.11d)$$

where $\xi^{i\alpha}$ and $\bar\xi^{\dot\alpha}_i$ are constant Weyl spinors with

$$\sigma(\xi^{i\alpha}) = \epsilon^{\alpha\beta}T^i_j\xi^{j\beta} \qquad \text{and} \qquad \sigma(\bar\xi^{\dot\alpha}_i) = \epsilon^{\dot\alpha\dot\beta}T^j_i\bar\xi^{\dot\beta}_j\,, \quad (2.12)$$

which is an immediate consequence of (2.6c) and (2.6d).

3. Deformed superspace $\mathbb{R}^{(4|4\mathcal{N})}_\hbar$

3.1 Definition of $\mathbb{R}^{(4|4\mathcal{N})}$

Before we start to discuss deformed superspaces, we must say a few words about graded Poisson structures. Let $\mathfrak{V}$ be a vector space over $\mathbb{R}$ or $\mathbb{C}$ equipped with a $\mathbb{Z}_2$-grading, i.e., $\mathfrak{V}$ is decomposed as $\mathfrak{V}\cong\bigoplus_{p=0,1}\mathfrak{V}_p$. Thus, elements of $\mathfrak{V}$ have either even ($p=0$) or odd ($p=1$) parity. Moreover, we can lift the vector space $\mathfrak{V}$ to a graded algebra by endowing it with an associative product which is assumed to respect the grading, i.e., for $f,g\in\mathfrak{V}$ and $\cdot: f\otimes g\mapsto f\cdot g$ we write

$$p(f\cdot g) = p(f)+p(g)g \mod 2\,.$$

By introducing a graded Lie bracket $[\ ,\ \}:\mathfrak{V}\otimes\mathfrak{V}\to\mathfrak{V}$ defined by

$$[f,g\} \equiv f\cdot g-(-)^{p_f p_g}g\cdot f\,, \quad (3.1)$$

the graded algebra $\mathfrak{V}$ becomes a graded Lie algebra. In the following we shall omit "$\cdot$". Of course, (3.1) satisfies a graded Jacobi identity,

$$[f,[g,h\}\}+(-)^{p_f(p_g+p_h)}[g,[h,f\}\}+(-)^{p_h(p_f+p_g)}[h,[g,f\}\} = 0\,, \quad (3.2)$$

for any $f,g,h\in\mathfrak{V}$. Then we define the superspace $\mathbb{R}^{(4|4\mathcal{N})}$ by

$$\mathbb{R}^{(4|4\mathcal{N})} \equiv C^\infty(\mathbb{R}^4)\otimes\Lambda^\bullet(\mathbb{R}^{4\mathcal{N}})\,, \quad (3.3)$$

where $\Lambda^\bullet(\mathbb{R}^{4\mathcal{N}})$ denotes the exterior algebra of $\mathbb{R}^{4\mathcal{N}}$. The elements of $\mathbb{R}^{(4|4\mathcal{N})}$ are called superfields. The algebra $\mathbb{R}^{(4|4\mathcal{N})}$ is finitely generated by $(X^a) = (x^{\alpha\dot\alpha},\theta^{i\alpha},\bar\theta^{\dot\alpha}_i)$ with

$\alpha, \beta, \ldots, \dot{\alpha}, \dot{\beta}, \ldots = 1,2$ and $i, j, \ldots = 1, \ldots, \mathcal{N}$. Indices $a, b, \ldots$ represent all generators. To these generators[4] we assign the following parities:

$$p_{x^{\alpha\dot{\alpha}}} \equiv 0 \qquad \text{and} \qquad p_{\theta^{i\alpha}} \equiv p_{\bar{\theta}_i^{\dot{\alpha}}} \equiv 1\,.$$

The action of σ is given by

$$\sigma(x^{\alpha\dot{\beta}}) = \epsilon^{\alpha\beta} x^{\beta\dot{\gamma}} \epsilon^{\dot{\beta}\dot{\gamma}}\,, \qquad \sigma(\theta^{i\alpha}) = \epsilon^{\alpha\beta} T^i_j \theta^{j\beta} \qquad \text{and} \qquad \sigma(\bar{\theta}_i^{\dot{\alpha}}) = \epsilon^{\dot{\alpha}\dot{\beta}} T_i^j \bar{\theta}_j^{\dot{\beta}}\,, \tag{3.4}$$

where T^i_j is given by (2.6e). Any superfield $f \in \mathbb{R}^{(4|4\mathcal{N})}$ may be expanded in terms of the $\theta^{i\alpha}$ and $\bar{\theta}_i^{\dot{\alpha}}$ coordinates as

$$f = \overset{\circ}{f}(x) + \sum_{0<|I|,|\bar{J}|\leq 2\mathcal{N}} f_{I,\bar{J}}(x) \theta^I \bar{\theta}^{\bar{J}}\,, \tag{3.5}$$

where I and $\bar{J}$ are multiindices with $I = (i_1\alpha_1, \ldots, i_{|I|}\alpha_{|I|})$ and $\bar{J} = (\,{}^{\dot{\alpha}_1}_{i_1}, \ldots, {}^{\dot{\alpha}_{|\bar{J}|}}_{i_{|\bar{J}|}})$. In (3.5) we suppressed the wedge symbol.

On $\mathbb{R}^{(4|4\mathcal{N})}$ we may introduce left and right derivations. A left derivation is a linear map $\overrightarrow{\partial}$ which satisfies

$$\overrightarrow{\partial}(fg) = \overrightarrow{\partial}(f)g + (-)^{p_\partial p_f} f \overrightarrow{\partial}(g)$$

for $f, g \in \mathbb{R}^{(4|4\mathcal{N})}$. Similarly, we may introduce right derivations as

$$f \overleftarrow{\partial} \equiv (-)^{p_\partial(p_f+1)} \overrightarrow{\partial} f\,.$$

In these equations p_∂ is called the degree of the derivation.

A graded (or super) Poisson structure on $\mathbb{R}^{(4|4\mathcal{N})}$ is a graded Lie algebra structure (3.1), which satisfies the Jacobi identity (3.2) and

$$\begin{aligned} [f, gh\} &= [f, g\}h + (-)^{p_f p_g} g[f, h\}, \\ [fg, h\} &= f[g, h\} + (-)^{p_g p_h}[f, h\}g, \end{aligned} \tag{3.6}$$

for $f, g, h \in \mathbb{R}^{(4|4\mathcal{N})}$.

3.2 Superderivatives, supercharges and supersymmetry algebra

Let $f \in \mathbb{R}^{(4|4\mathcal{N})}$. The left superderivatives are defined in the usual way as

$$\begin{aligned} \overrightarrow{D}_{i\alpha} f &\equiv \overrightarrow{\partial}_{i\alpha} f + \bar{\theta}_i^{\dot{\alpha}} \partial_{\alpha\dot{\alpha}} f\,, \\ \overrightarrow{\bar{D}}{}^i_{\dot{\alpha}} f &\equiv -\overrightarrow{\bar{\partial}}{}^i_{\dot{\alpha}} f - \theta^{i\alpha} \partial_{\alpha\dot{\alpha}} f\,. \end{aligned} \tag{3.7}$$

They satisfy the following algebra:

$$\left\{\overrightarrow{D}_{i\alpha}, \overrightarrow{D}_{j\beta}\right\} = 0\,, \qquad \left\{\overrightarrow{\bar{D}}{}^i_{\dot{\alpha}}, \overrightarrow{\bar{D}}{}^j_{\dot{\beta}}\right\} = 0 \qquad \text{and} \qquad \left\{\overrightarrow{D}_{i\alpha}\,, \overrightarrow{\bar{D}}{}^j_{\dot{\beta}}\right\} = -2\delta^i_j \partial_{\alpha\dot{\beta}}\,. \tag{3.8}$$

The definition of the right superderivatives $\overleftarrow{D}_{i\alpha}$ and $\overleftarrow{\bar{D}}{}^i_{\dot{\alpha}}$ is then immediate.

[4] In the following we shall call them loosely "coordinates".

The left supercharge operators are given by the expressions

$$\begin{aligned}\overrightarrow{Q}_{i\alpha} f &\equiv \overrightarrow{\partial}_{i\alpha} f - \bar{\theta}^{\dot{\alpha}}_i \partial_{\alpha\dot{\alpha}} f\,, \\ \overrightarrow{Q}^i_{\dot{\alpha}} f &\equiv -\overrightarrow{\partial}^i_{\dot{\alpha}} f + \theta^{i\alpha} \partial_{\alpha\dot{\alpha}} f\,.\end{aligned} \tag{3.9}$$

They obviously satisfy

$$\left\{\overrightarrow{Q}_{i\alpha}, \overrightarrow{Q}_{j\beta}\right\} = 0\,, \qquad \left\{\overrightarrow{Q}^i_{\dot{\alpha}}, \overrightarrow{Q}^j_{\dot{\beta}}\right\} = 0 \qquad \text{and} \qquad \left\{\overrightarrow{Q}_{i\alpha}, \overrightarrow{Q}^j_{\dot{\beta}}\right\} = 2\delta^i_j \partial_{\alpha\dot{\beta}} \tag{3.10}$$

and they anticommute with the left superderivatives (3.7). Recall that the supercharges (3.9) generate the following (super)translations on the superspace

$$\begin{aligned}\theta^{i\alpha} &\mapsto \theta'^{\,i\alpha} = \theta^{i\alpha} + \xi^{i\alpha}\,, \\ \bar{\theta}^{\dot{\alpha}}_i &\mapsto \bar{\theta}'^{\dot{\alpha}}_i = \bar{\theta}^{\dot{\alpha}}_i + \bar{\xi}^{\dot{\alpha}}_i\,, \\ x^{\alpha\dot{\alpha}} &\mapsto x'^{\alpha\dot{\alpha}} = x^{\alpha\dot{\alpha}} + a^{\alpha\dot{\alpha}} - (\xi^{i\alpha}\bar{\theta}^{\dot{\alpha}}_i + \bar{\xi}^{\dot{\alpha}}_i \theta^{i\alpha})\,,\end{aligned} \tag{3.11}$$

where $\xi^{i\alpha}$ and $\bar{\xi}^{\dot{\alpha}}_i$ are constant Majorana-Weyl spinors and $a^{\alpha\dot{\alpha}}$ represents a constant four-vector.

3.3 Definition of $\mathbb{R}_\hbar^{(4|4\mathcal{N})}$

Having fixed the undeformed setup we are now ready for the discussion of a non(anti)commutative extension of our theory.

Let $[\ ,\ \}$ be a super Poisson structure. Moreover, we define an operator $P : \mathbb{R}^{(4|4\mathcal{N})} \otimes \mathbb{R}^{(4|4\mathcal{N})} \to \mathbb{R}^{(4|4\mathcal{N})}$ by

$$P(f \otimes g) \equiv [f, g\}\,.$$

Now we consider the algebra $\mathbb{R}^{(4|4\mathcal{N})}[[\hbar]]$ which we abbreviate by $\mathbb{R}_\hbar^{(4|4\mathcal{N})}$ in the sequel. The variable $\hbar$ is some parameter in which we consider a formal power series expansion. Of course, in the limit $\hbar \to 0$ we recover $\mathbb{R}^{(4|4\mathcal{N})}$. Then we define a star product by

$$\begin{aligned}\star : \mathbb{R}_\hbar^{(4|4\mathcal{N})} \otimes \mathbb{R}_\hbar^{(4|4\mathcal{N})} &\to \mathbb{R}_\hbar^{(4|4\mathcal{N})}\,, \\ f \otimes g \mapsto e^{\hbar P}(f \otimes g) = \sum_n \frac{\hbar^n}{n!} P^n(f \otimes g) \ &= \ fg + \hbar[f, g\} + \mathcal{O}(\hbar^2)\,.\end{aligned} \tag{3.12}$$

In this paper we assume that P is a bi-differential operator of the form

$$P^n(f \otimes g) = \frac{1}{2^n} C^{i_1\alpha_1, j_1\beta_1} \cdots C^{i_n\alpha_n, j_n\beta_n} f \overleftarrow{Q}_{i_1\alpha_1} \cdots \overleftarrow{Q}_{i_n\alpha_n} \overrightarrow{Q}_{j_n\beta_n} \cdots \overrightarrow{Q}_{j_1\beta_1} g\,, \tag{3.13}$$

where $C^{i\alpha, j\beta} \in \mathrm{Sym}(2\mathcal{N}, \mathbb{C})$, i.e., $i, j = 1, \ldots, \mathcal{N}$. Moreover, $C^{i\alpha, j\beta}$ is assumed to be constant. Note that in this way we have defined an associative and nilpotent star product, i.e., the series expansion in (3.12) is finite and goes up to $\mathcal{O}(\hbar^{2\mathcal{N}})$.

Equation (3.13) implies that

$$f \star g = f \exp\left\{\frac{\hbar}{2} \overleftarrow{Q}_{i\alpha} C^{i\alpha, j\beta} \overrightarrow{Q}_{j\beta}\right\} g\,. \tag{3.14}$$

Using the definition (3.7), one may readily verify that

$$\begin{aligned} x^{\alpha\dot\alpha}\star x^{\beta\dot\beta} &= x^{\alpha\dot\alpha}x^{\beta\dot\beta}-\frac{\hbar}{2}C^{i\alpha,j\beta}\bar\theta_i^{\dot\alpha}\bar\theta_j^{\dot\beta}\,,\\ x^{\alpha\dot\alpha}\star\theta^{j\beta} &= x^{\alpha\dot\alpha}\theta^{j\beta}+\frac{\hbar}{2}C^{i\alpha,j\beta}\bar\theta_i^{\dot\alpha}\,,\\ \theta^{i\alpha}\star\theta^{j\beta} &= \theta^{i\alpha}\theta^{j\beta}+\frac{\hbar}{2}C^{i\alpha,j\beta}\,. \end{aligned}\tag{3.15}$$

Therefore we may introduce a star supercommutator by

$$[f,g\}_\star\equiv f\star g-(-)^{p_fp_g}g\star f \tag{3.16}$$

leading to

$$\begin{aligned} [x^{\alpha\dot\alpha},x^{\beta\dot\beta}]_\star &= -\hbar\,C^{i\alpha,j\beta}\bar\theta_i^{\dot\alpha}\bar\theta_j^{\dot\beta}\,,\\ [x^{\alpha\dot\alpha},\theta^{j\beta}]_\star &= \hbar\,C^{i\alpha,j\beta}\bar\theta_i^{\dot\alpha}\,,\\ \{\theta^{i\alpha},\theta^{j\beta}\}_\star &= \hbar\,C^{i\alpha,j\beta}\,. \end{aligned}\tag{3.17}$$

Of course, now we also have a star Jacobi identity

$$[f,[g,h\}_\star\}_\star+(-)^{p_f(p_g+p_h)}[g,[h,f\}_\star\}_\star+(-)^{p_h(p_f+p_g)}[h,[g,f\}_\star\}_\star=0\,, \tag{3.18}$$

for $f,g,h\in\mathbb{R}_\hbar^{(4|4\mathcal{N})}$.

Obviously, the algebra (3.17) does not transform covariantly under the full set of (super)translations (3.11). Depending on the rank of the deformation matrix $C^{i\alpha,j\beta}$, the supersymmetry will partially be broken. Note that here we are assuming an undeformed parameter algebra, i.e., $\epsilon^{i\alpha}$, $\bar\epsilon_i^{\dot\alpha}$ and $a^{\alpha\dot\alpha}$ are kept (anti)commuting.

3.4 Deformed supersymmetry algebra

As it will become clear momentarily, it will be convenient for us to use chiral coordinates on $\mathbb{R}_\hbar^{(4|4\mathcal{N})}$ instead of $(X^a)=(x^{\alpha\dot\alpha},\theta^{i\alpha},\bar\theta_i^{\dot\alpha})$. These are defined by[5]

$$(X^a)=(x^{\alpha\dot\alpha},\theta^{i\alpha},\bar\theta_i^{\dot\alpha})\mapsto(Y^a)=(y^{\alpha\dot\alpha}=x^{\alpha\dot\alpha}+\theta^{i\alpha}\bar\theta_i^{\dot\alpha},\theta^{i\alpha},\bar\theta_i^{\dot\alpha})\,. \tag{3.19}$$

It is easy to check that the involution σ acts as

$$\sigma(y^{\alpha\dot\beta})=\epsilon^{\alpha\beta}y^{\beta\dot\gamma}\epsilon^{\dot\beta\dot\gamma}\,. \tag{3.20}$$

In the coordinates (3.19) the superderivatives (3.7) and the supercharges (3.9) take the following form

$$\overrightarrow{D}_{i\alpha}=\overrightarrow{\partial}_{i\alpha}+2\bar\theta_i^{\dot\alpha}\partial_{\alpha\dot\alpha}\qquad\text{and}\qquad\overrightarrow{D}^i_{\dot\alpha}=-\overrightarrow{\partial}^i_{\dot\alpha}\,, \tag{3.21a}$$

$$\overrightarrow{Q}_{i\alpha}=\overrightarrow{\partial}_{i\alpha}\qquad\text{and}\qquad\overrightarrow{Q}^i_{\dot\alpha}=-\overrightarrow{\partial}^i_{\dot\alpha}+2\theta^{i\alpha}\partial_{\alpha\dot\alpha}\,, \tag{3.21b}$$

where now the $\partial_{\alpha\dot\alpha}$s are partial derivatives with respect to $y^{\alpha\dot\alpha}$.

[5]Note that $\theta^{i\alpha}\star\bar\theta_i^{\dot\alpha}=\theta^{i\alpha}\bar\theta_i^{\dot\alpha}$.

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One may readily check that the (anti)commutation relations (3.17) become

$$
\begin{aligned}
[y^{\alpha\dot\alpha}, y^{\beta\dot\beta}]_\star &= 0,\\
[y^{\alpha\dot\alpha}, \theta^{j\beta}]_\star &= 0,\\
\{\theta^{i\alpha}, \theta^{j\beta}\}_\star &= \hbar\, C^{i\alpha,j\beta}\,.
\end{aligned}
\tag{3.22}
$$

Furthermore, it is obvious from the explicit form (3.21a) of the superderivatives that they satisfy

$$
\begin{aligned}
\{\overrightarrow{D}_{i\alpha}, \overrightarrow{D}_{j\beta}\}_\star &= 0\,,\\
\{\overrightarrow{D}^i_{\dot\alpha}, \overrightarrow{D}^j_{\dot\beta}\}_\star &= 0\,,\\
\{\overrightarrow{D}_{i\alpha}, \overrightarrow{D}^j_{\dot\beta}\}_\star &= -2\delta^i_j \partial_{\alpha\dot\beta}\,,
\end{aligned}
\tag{3.23}
$$

i.e., the same algebra as in the undeformed case. The supercharge operators are subject to the following relations

$$
\begin{aligned}
\{\overrightarrow{Q}_{i\alpha}, \overrightarrow{Q}_{j\beta}\}_\star &= 0,\\
\{\overrightarrow{Q}^i_{\dot\alpha}, \overrightarrow{Q}^j_{\dot\beta}\}_\star &= 4\hbar\, C^{i\alpha,j\beta}\partial_{\alpha\dot\alpha}\partial_{\beta\dot\beta}\,,\\
\{\overrightarrow{Q}_{i\alpha}, \overrightarrow{Q}^j_{\dot\beta}\}_\star &= 2\delta^i_j \partial_{\alpha\dot\beta}\,,
\end{aligned}
\tag{3.24}
$$

while the star anticommutators between the superderivatives and supercharges do still vanish. In the sequel, we shall refer to (3.23) and (3.24) as the deformed supersymmetry algebra. The explicit form of algebra (3.24) makes the supersymmetry breaking apparent. Note that the change of coordinates (3.19) was needed since otherwise the $\overrightarrow{D}^i_{\dot\alpha}$s would not be derivations with respect to the star product. This property, however, is essential in the subsequent discussion.

4. Constraint equations and field equations on $\mathbb{R}^{(4|16)}_\hbar$

Now we are discussing the $\mathcal{N}=4$ case. The goal of this section is to derive the field equations of deformed $\mathcal{N}=4$ super Yang-Mills theory from the constraint equations. The discussion in the undeformed case is given in [26, 27].

In the sequel, let $\mathfrak{g}$ be the gauge algebra. For instance, we could take $\mathfrak{g}$ to be $\mathfrak{u}(n)$. Generally speaking, $\mathfrak{g}$ is some enveloping algebra. In order to simplify notation, we shall omit the arrows over the derivatives. If not stated differently, all derivatives are left derivatives.

4.1 Superfield equations from constraint equations

From now on we shall adopt the common convention and write always hats when we mean deformed superfields.

Starting point is the deformed constraint equations

$$
\{\hat\nabla_{i\alpha}, \hat\nabla_{j\beta}\}_\star = -2\epsilon_{\alpha\beta}\hat W_{ij}\,,
\tag{4.1a}
$$

$$\{\hat{\bar{\nabla}}^i_{\dot\alpha}, \hat{\bar{\nabla}}^j_{\dot\beta}\}_\star = -\epsilon_{\dot\alpha\dot\beta}\epsilon^{ijkl}\hat{W}_{kl}\,, \tag{4.1b}$$

$$\{\hat{\nabla}_{i\alpha}, \hat{\bar{\nabla}}^j_{\dot\beta}\}_\star = -2\delta^j_i \hat{\nabla}_{\alpha\dot\beta}\,. \tag{4.1c}$$

The covariant derivatives in (4.1) are given by

$$\hat{\nabla}_{i\alpha} = D_{i\alpha} + [\hat\omega_{i\alpha},\ \}_\star\,, \qquad \hat{\bar{\nabla}}^i_{\dot\alpha} = \bar{D}^i_{\dot\alpha} - [\hat{\bar\omega}^i_{\dot\alpha},\ \}_\star \qquad \text{and} \qquad \hat{\nabla}_{\alpha\dot\beta} = \partial_{\alpha\dot\beta} + [\hat{A}_{\alpha\dot\beta},\ ]_\star\,. \tag{4.2}$$

More explicitly, the equations (4.1) read as

$$D_{i\alpha}\hat\omega_{j\beta} + D_{j\beta}\hat\omega_{i\alpha} + \{\hat\omega_{i\alpha}, \hat\omega_{j\beta}\}_\star = -2\epsilon_{\alpha\beta}\hat{W}_{ij}\,, \tag{4.3a}$$

$$\bar{D}^i_{\dot\alpha}\hat{\bar\omega}^j_{\dot\beta} + \bar{D}^j_{\dot\beta}\hat{\bar\omega}^i_{\dot\alpha} - \{\hat{\bar\omega}^i_{\dot\alpha}, \hat{\bar\omega}^j_{\dot\beta}\}_\star = \epsilon_{\dot\alpha\dot\beta}\epsilon^{ijkl}\hat{W}_{kl}, \tag{4.3b}$$

$$D_{i\alpha}\hat{\bar\omega}^j_{\dot\beta} - \bar{D}^j_{\dot\beta}\hat\omega_{i\alpha} + \{\hat\omega_{i\alpha}, \hat{\bar\omega}^j_{\dot\beta}\}_\star = 2\delta^j_i\ \hat{A}_{\alpha\dot\beta}\,. \tag{4.3c}$$

As usual, we decompose the bosonic curvature as

$$[\hat\nabla_{\alpha\dot\alpha}, \hat\nabla_{\beta\dot\beta}]_\star = \epsilon_{\dot\alpha\dot\beta}\hat{f}_{\alpha\beta} + \epsilon_{\alpha\beta}\hat{f}_{\dot\alpha\dot\beta}\,. \tag{4.4}$$

Then we define the superspinor fields by the equations

$$[\hat\nabla_{i\alpha}, \hat\nabla_{\beta\dot\beta}]_\star \equiv \epsilon_{\alpha\beta}\hat{\bar\chi}_{i\dot\beta}\,, \tag{4.5a}$$

$$[\hat{\bar\nabla}^i_{\dot\alpha}, \hat\nabla_{\beta\dot\beta}]_\star \equiv \epsilon_{\dot\alpha\dot\beta}\hat\chi^i_\beta\,, \tag{4.5b}$$

which are

$$D_{i\alpha}\hat{A}_{\beta\dot\beta} - \partial_{\beta\dot\beta}\hat\omega_{i\alpha} + [\hat\omega_{i\alpha}, \hat{A}_{\beta\dot\beta}]_\star = \epsilon_{\alpha\beta}\hat{\bar\chi}_{i\dot\beta}\,, \tag{4.6a}$$

$$\bar{D}^i_{\dot\alpha}\hat{A}_{\beta\dot\beta} + \partial_{\beta\dot\beta}\hat{\bar\omega}^i_{\dot\alpha} - [\hat{\bar\omega}^i_{\dot\alpha}, \hat{A}_{\beta\dot\beta}]_\star = \epsilon_{\dot\alpha\dot\beta}\hat\chi^i_\beta\,. \tag{4.6b}$$

As an immediate consequence of the graded Bianchi identities,

$$[\hat\nabla_a, [\hat\nabla_b, \hat\nabla_c\}_\star\}_\star + (-)^{p_a(p_b+p_c)}[\hat\nabla_b, [\hat\nabla_c, \hat\nabla_a\}_\star\}_\star + (-)^{p_c(p_a+p_b)}[\hat\nabla_c, [\hat\nabla_a, \hat\nabla_b\}_\star\}_\star = 0\,,$$

where $\hat\nabla_a$ is either $\hat\nabla_{\alpha\dot\alpha}$, $\hat\nabla_{i\alpha}$ or $\hat{\bar\nabla}^i_{\dot\alpha}$, we have

$$\hat\nabla_{i\alpha}\hat{W}_{jk} = \epsilon_{ijkl}\hat\chi^l_\alpha\,, \tag{4.7a}$$

$$\hat{\bar\nabla}^i_{\dot\alpha}\hat{W}_{jk} = \delta^i_j\hat{\bar\chi}_{k\dot\alpha} - \delta^i_k\hat{\bar\chi}_{j\dot\alpha}\,, \tag{4.7b}$$

$$\hat\nabla_{i\alpha}\hat{\bar\chi}_{j\dot\alpha} = -2\hat\nabla_{\alpha\dot\alpha}\hat{W}_{ij}\,, \tag{4.7c}$$

$$\hat{\bar\nabla}^i_{\dot\alpha}\hat\chi^j_\alpha = -\epsilon^{ijkl}\hat\nabla_{\alpha\dot\alpha}\hat{W}_{kl}\,. \tag{4.7d}$$

Considering the Bianchi identities for $(\hat\nabla_{i\alpha}, \hat{\bar\nabla}^j_{\dot\beta}, \hat\nabla_{\gamma\dot\gamma})$ together with the Bianchi identities for $(\hat\nabla_{i\alpha}, \hat\nabla_{j\beta}, \hat{W}_{lk})$ and equation (4.4), we discover

$$\hat\nabla_{i\alpha}\hat\chi^j_\beta = -2\delta^j_i\hat{f}_{\alpha\beta} - \frac{1}{2}\epsilon_{\alpha\beta}\epsilon^{jklm}[\hat{W}_{lm}, \hat{W}_{ik}]_\star\,, \tag{4.8a}$$

$$\hat{\bar\nabla}^j_{\dot\alpha}\hat{\bar\chi}_{i\dot\beta} = -2\delta^j_i\hat{f}_{\dot\alpha\dot\beta} + \frac{1}{2}\epsilon_{\dot\alpha\dot\beta}\epsilon^{jklm}[\hat{W}_{lm}, \hat{W}_{ik}]_\star\,. \tag{4.8b}$$

Similarly, again by using Bianchi identities a straightforward computation yields

$$\hat{\nabla}_{i\alpha}\hat{f}_{\beta\gamma} = -\frac{1}{2}\epsilon^{\dot{\beta}\dot{\gamma}}(\epsilon_{\alpha\beta}\hat{\nabla}_{\gamma\dot{\beta}}\hat{\bar{\chi}}_{i\dot{\gamma}} + \epsilon_{\alpha\gamma}\hat{\nabla}_{\beta\dot{\beta}}\hat{\bar{\chi}}_{i\dot{\gamma}})\,, \tag{4.9a}$$

$$\hat{\bar{\nabla}}^{i}_{\dot{\alpha}}\hat{f}_{\beta\gamma} = \frac{1}{2}(\hat{\nabla}_{\beta\dot{\alpha}}\hat{\chi}^{i}_{\gamma} + \hat{\nabla}_{\gamma\dot{\alpha}}\hat{\chi}^{i}_{\beta})\,, \tag{4.9b}$$

$$\hat{\nabla}_{i\alpha}\hat{f}_{\dot{\beta}\dot{\gamma}} = \frac{1}{2}(\hat{\nabla}_{\alpha\dot{\beta}}\hat{\bar{\chi}}_{i\dot{\gamma}} + \hat{\nabla}_{\alpha\dot{\gamma}}\hat{\bar{\chi}}_{i\dot{\beta}})\,, \tag{4.9c}$$

$$\hat{\bar{\nabla}}^{i}_{\dot{\alpha}}\hat{f}_{\dot{\beta}\dot{\gamma}} = -\frac{1}{2}\epsilon^{\beta\gamma}(\epsilon_{\dot{\alpha}\dot{\beta}}\hat{\nabla}_{\gamma\dot{\gamma}}\hat{\chi}^{i}_{\beta} + \epsilon_{\dot{\alpha}\dot{\gamma}}\hat{\nabla}_{\gamma\dot{\beta}}\hat{\chi}^{i}_{\beta})\,. \tag{4.9d}$$

Furthermore, from equation (4.1c) we deduce

$$\hat{\nabla}_{\alpha\dot{\alpha}} = -\frac{1}{8}\{\hat{\nabla}_{i\alpha}, \hat{\bar{\nabla}}^{i}_{\dot{\alpha}}\}_{\star}\,. \tag{4.10}$$

Applying it to $\hat{\chi}^{i}_{\beta}$ and $\hat{\bar{\chi}}_{i\dot{\beta}}$ and using the equations (4.5), (4.7) and (4.8) we obtain the super Dirac equations

$$\epsilon^{\alpha\beta}\hat{\nabla}_{\alpha\dot{\alpha}}\hat{\chi}^{i}_{\beta} + \frac{1}{2}\epsilon^{ijkl}[\hat{W}_{kl}, \hat{\bar{\chi}}_{j\dot{\alpha}}]_{\star} = 0\,, \tag{4.11a}$$

$$\epsilon^{\dot{\alpha}\dot{\beta}}\hat{\nabla}_{\alpha\dot{\alpha}}\hat{\bar{\chi}}_{i\dot{\beta}} + [\hat{W}_{ij}, \hat{\chi}^{j}_{\alpha}]_{\star} = 0\,. \tag{4.11b}$$

Note that they look formally the same as in the undeformed setup, but this time all products are replaced by star products.

Applying $\hat{\nabla}_{m\gamma}$ to (4.11a) and to (4.11b) and using the equations (4.5), (4.7) and (4.8) we get the superfield equations of motion for the gauge field and the scalar multiplet

$$\epsilon^{\dot{\alpha}\dot{\beta}}\hat{\nabla}_{\gamma\dot{\alpha}}\hat{f}_{\dot{\beta}\dot{\gamma}} + \epsilon^{\alpha\beta}\hat{\nabla}_{\alpha\dot{\gamma}}\hat{f}_{\beta\gamma} = \frac{1}{4}\epsilon^{ijkl}[\hat{\nabla}_{\gamma\dot{\gamma}}\hat{W}_{ij}, \hat{W}_{kl}]_{\star} + \{\hat{\chi}^{i}_{\gamma}, \hat{\bar{\chi}}_{i\dot{\gamma}}\}_{\star}, \tag{4.12a}$$

$$\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}\hat{\nabla}_{\alpha\dot{\alpha}}\hat{\nabla}_{\beta\dot{\beta}}\hat{W}_{ij} - \frac{1}{4}\epsilon^{klmn}[\hat{W}_{mn}, [\hat{W}_{kl}, \hat{W}_{ij}]_{\star}]_{\star} = \frac{1}{2}\epsilon_{ijkl}\epsilon^{\alpha\beta}\{\hat{\chi}^{k}_{\alpha}, \hat{\chi}^{l}_{\beta}\}_{\star} + \epsilon^{\dot{\alpha}\dot{\beta}}\{\hat{\bar{\chi}}_{i\dot{\alpha}}, \hat{\bar{\chi}}_{j\dot{\beta}}\}_{\star}. \tag{4.12b}$$

4.2 Superfield expansions

In the previous subsection, we have derived the superfield equations of motion from the constraint equations on $\mathbb{R}^{(4|16)}_{\hbar}$. Now we are interested in the superfield expansions. Of particular interest is, of course, the zeroth order components of (4.11) and (4.12), i.e., those terms which do not contain the odd coordinates $\theta^{i\alpha}$ and $\bar{\theta}^{\dot{\alpha}}_{i}$.

The problem we are faced with is to find a proper way to construct the deformed superfields which, as we have already seen in [12], will differ from their undeformed pendants. In the appendix A, we review their construction in the undeformed case.

One possibility is to find some proper recursion operator as was done in the undeformed setup (see references [26, 27]). Following this approach, one encounters the difficulty that all theta orders of the superfields do mix, i.e., one discovers a coupled non-linear system of algebraic equations for the component fields, which one has to solve simultaneously for all the components. However, already in the undeformed setup the superfield expansions become quite lengthy as one considers higher and higher powers in the odd coordinates. Therefore

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we suggest to follow another way which is based on a generalization of the Seiberg-Witten map [28] to superspace.[6] Eventually, this yields a systematic way to construct the deformed superfield equations order by order in the deformation $\hbar$. Such a generalization of the Seiberg-Witten map seems quite natural and has been conjectured throughout the literature [10, 12]. We shall add some remarks on this topic in the conclusions.

Remember that we have one *fundamental* field in our theory. For instance, we may regard the gauge potential $\hat{\omega}_{i\alpha}$ as fundamental. This simply means that all other field expansions, i.e., those for $\hat{\bar{\omega}}^i_{\dot\alpha}$, $\hat{A}_{\alpha\dot\alpha}$, $\hat{W}_{ij}$, $\hat{\chi}^i_\alpha$ and $\hat{\bar{\chi}}_{i\dot\alpha}$ are completely determined through $\hat{\omega}_{i\alpha}$ via the constraint equations (4.1) and the definitions (4.5) (in a certain gauge; see below).

As in ordinary noncommutative field theory, the starting point is then the equation

$$\hat{\omega}_{i\alpha}(\omega + \delta_\lambda\omega, \bar{\omega} + \delta_\lambda\bar{\omega}) = \hat{\omega}_{i\alpha}(\omega, \bar{\omega}) + \delta_{\hat{\Lambda}}\hat{\omega}_{i\alpha}(\omega, \bar{\omega})\,. \tag{4.13}$$

Recall that infinitesimal gauge transformations of the undeformed superfield $\omega_{i\alpha}$ is induced by an even superfield λ via

$$\delta_\lambda \omega_{i\alpha} = D_{i\alpha}\lambda + [\omega_{i\alpha}, \lambda]\,, \tag{4.14}$$

whereas for the deformed superfield we have

$$\delta_{\hat{\Lambda}}\hat{\omega}_{i\alpha} = D_{i\alpha}\hat{\Lambda} + [\hat{\omega}_{i\alpha}, \hat{\Lambda}]_\star\,. \tag{4.15}$$

Instead of directly solving (4.13) for the gauge potential and the gauge parameter, we may consider a consistency condition of two successive gauge transformations (see, e.g., [33] and references therein). Take, for instance, some superfield $\hat{\psi}$ which transforms in the fundamental representation of the gauge group, i.e.,

$$\delta_{\hat{\Lambda}}\hat{\psi} = -\hat{\Lambda} \star \hat{\psi} \tag{4.16}$$

and hence

$$[\delta_{\hat{\Lambda}}, \delta_{\hat{\Sigma}}]\hat{\psi} = -[\hat{\Lambda}, \hat{\Sigma}]_\star \star \hat{\psi} = \delta_{[\hat{\Lambda},\hat{\Sigma}]_\star}\hat{\psi}\,. \tag{4.17}$$

Following [33], we are looking for gauge transformations of the type

$$\hat{\Lambda}(\lambda, \omega, \bar{\omega}) \equiv \hat{\Lambda}_\lambda(\omega, \bar{\omega})\,.$$

Therefore we restrict the transformation (4.16) to

$$\delta_\lambda\hat{\psi} = -\hat{\Lambda}_\lambda(\omega, \bar{\omega}) \star \hat{\psi}\,. \tag{4.18}$$

Thus, equation (4.17) translates into

$$\delta_\lambda\hat{\Lambda}_\sigma - \delta_\sigma\hat{\Lambda}_\lambda + [\hat{\Lambda}_\lambda, \hat{\Lambda}_\sigma]_\star = \hat{\Lambda}_{[\lambda,\sigma]}\,. \tag{4.19}$$

Now we assume that it is possible to expand $\hat{\Lambda}_\lambda$ in powers of $\hbar$, i.e., we write

$$\hat{\Lambda}_\lambda = \lambda + \hbar\,\hat{\Lambda}^1_\lambda + \mathcal{O}(\hbar^2)\,. \tag{4.20}$$

[6] In the case of purely bosonic deformations, Seiberg-Witten maps for $\mathcal{N} = 1$ superfields have been discussed in [31, 32].

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Expanding equation (4.19) in powers of $\hbar$ and substituting equation (4.20), we realize that to zeroth order it is trivially satisfied. To first order in $\hbar$ we obtain

$$\delta_\lambda \hat{\Lambda}^1_\sigma - \delta_\sigma \hat{\Lambda}^1_\lambda + [\lambda, \hat{\Lambda}^1_\sigma] + [\hat{\Lambda}^1_\lambda, \sigma] - \frac{1}{2} C^{i\alpha,j\beta} [\partial_{i\alpha}\lambda, \partial_{j\beta}\sigma] = \hat{\Lambda}^1_{[\lambda,\sigma]} \,.$$

A solution to this equation is of the form

$$\hat{\Lambda}^1_\lambda = -\frac{1}{4} C^{i\alpha,j\beta} [\partial_{i\alpha}\lambda, \Omega_{j\beta}] \,, \tag{4.21}$$

where we have introduced

$$\Omega_{i\alpha} \equiv \omega_{i\alpha} + \bar{\theta}^{\dot\beta}_j (\bar{D}^j_{\dot\beta}\omega_{i\alpha} + D_{i\alpha}\bar{\omega}^j_{\dot\beta} + \{\bar{\omega}^j_{\dot\beta}, \omega_{i\alpha}\}) \,. \tag{4.22}$$

In order to verify (4.21), we note that infinitesimal gauge transformations act on $\Omega_{i\alpha}$ as

$$\delta_\lambda \Omega_{i\alpha} = \partial_{i\alpha}\lambda + [\Omega_{i\alpha}, \lambda] \,.$$

Therefore we may write

$$\hat{\Lambda}_\lambda = \lambda - \frac{\hbar}{4} C^{i\alpha,j\beta} [\partial_{i\alpha}\lambda, \Omega_{j\beta}] + \mathcal{O}(\hbar^2) \,. \tag{4.23}$$

Having derived the expansion for the gauge parameter to first order in $\hbar$, it is now easy to give the expansion for the super gauge potential $\hat{\omega}_{i\alpha}$. Consider the expansion

$$\hat{\omega}_{i\alpha} = \omega_{i\alpha} + \hbar\, \hat{\omega}^1_{i\alpha} + \mathcal{O}(\hbar^2) \,. \tag{4.24}$$

Equation (4.15) then yields

$$\delta_\lambda \hat{\omega}^1_{i\alpha} = D_{i\alpha}\hat{\Lambda}^1_\lambda + [\hat{\omega}^1_{i\alpha}, \lambda] + [\omega_{i\alpha}, \hat{\Lambda}^1_\lambda] + \frac{1}{2} C^{j\beta,k\gamma} \{\partial_{j\beta}\omega_{i\alpha}, \partial_{k\gamma}\lambda\} \,.$$

Note that our equations are again satisfied identically to zeroth order. Substituting our solution (4.21) into the above equation, one finds after some algebraic manipulations that

$$\hat{\omega}^1_{i\alpha} = \frac{1}{4} C^{j\beta,k\gamma} \{\Omega_{j\beta}, \partial_{k\gamma}\omega_{i\alpha} + R_{k\gamma,i\alpha}\}, \tag{4.25}$$

with

$$R_{i\alpha,j\beta} \equiv \partial_{i\alpha}\omega_{j\beta} + D_{j\beta}\Omega_{i\alpha} + \{\omega_{j\beta}, \Omega_{i\alpha}\} \tag{4.26}$$

is a solution. Note that $R_{i\alpha,j\beta}$ transforms under infinitesimal gauge transformations as

$$\delta_\lambda R_{i\alpha,j\beta} = [R_{i\alpha,j\beta}, \lambda] \,.$$

Thus, we have

$$\hat{\omega}_{i\alpha} = \omega_{i\alpha} + \frac{\hbar}{4} C^{j\beta,k\gamma} \{\Omega_{j\beta}, \partial_{k\gamma}\omega_{i\alpha} + R_{k\gamma,i\alpha}\} + \mathcal{O}(\hbar^2) \,. \tag{4.27}$$

In order to find the field expansions for the remaining fields, we use the constraints (4.1) and the definitions (4.5). The expansion of (4.1a) to first order in $\hbar$ leads directly to

$$\hat{W}^1_{ij} = \frac{1}{2} \epsilon^{\alpha\beta} \nabla_{(i\alpha}\hat{\omega}^1_{j\beta)} + \frac{1}{8} \epsilon^{\alpha\beta} C^{m\delta,n\epsilon} \{\partial_{m\delta}\omega_{i\alpha}, \partial_{n\epsilon}\omega_{j\beta}\}, \tag{4.28}$$

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where the parentheses mean normalized symmetrization. This solution can be substituted into (4.1b) to give the first order contribution of the gauge potential $\hat{\bar{\omega}}^i_{\dot\alpha}$. Assuming that

$$\bar\nabla^i_{\dot\alpha}\hat{\bar{\omega}}^{j\,1}_{\dot\beta} = \bar\nabla^j_{\dot\beta}\hat{\bar{\omega}}^{i\,1}_{\dot\alpha}\,,$$

we arrive at the equation

$$\begin{aligned}\bar\nabla^i_{\dot\alpha}\hat{\bar{\omega}}^{j\,1}_{\dot\beta} &= \frac{1}{2}\epsilon_{\dot\alpha\dot\beta}\epsilon^{ijkl}\hat{W}^1_{kl} + \frac{1}{4}C^{m\delta,n\epsilon}\{\partial_{m\delta}\bar\omega^i_{\dot\alpha}, \partial_{n\epsilon}\bar\omega^j_{\dot\beta}\} \\ &= \bar{D}^i_{\dot\alpha}\hat{\bar{\omega}}^{j\,1}_{\dot\beta} - \{\bar\omega^i_{\dot\alpha}, \hat{\bar{\omega}}^{j\,1}_{\dot\beta}\}\,. \end{aligned} \tag{4.29}$$

Remember that in the undeformed setup one considers a certain gauge,[7] namely $\theta\omega - \bar\theta\bar\omega = \theta^{i\alpha}\omega_{i\alpha} + \bar\theta^{\dot\alpha}_i\bar\omega^i_{\dot\alpha} = 0$. In the deformed regime we want to choose a similar gauge,

$$\theta\hat\omega - \bar\theta\hat{\bar\omega} = \theta^{i\alpha}\hat\omega_{i\alpha} + \bar\theta^{\dot\alpha}_i\hat{\bar\omega}^i_{\dot\alpha} = 0\,, \tag{4.30}$$

implying that we have to all powers in $\hbar$

$$\theta^{i\alpha}\hat\omega^{(n)}_{i\alpha} + \bar\theta^{\dot\alpha}_i\hat{\bar\omega}^{i\,(n)}_{\dot\alpha} = 0\,. \tag{4.31}$$

Considering this equation for $n = 1$, one finds that the gauge potential has to satisfy the relation

$$\hat{\bar\omega}^{i\,1}_{\dot\alpha} = \bar{D}^i_{\dot\alpha}(\theta\hat\omega^1) - \bar\theta^{\dot\beta}_j\bar{D}^i_{\dot\alpha}\hat{\bar\omega}^{j\,1}_{\dot\beta}\,. \tag{4.32}$$

Substituting this equation into (4.29), we discover

$$\bar{D}^i_{\dot\alpha}\hat{\bar\omega}^{j\,1}_{\dot\beta} - \bar\theta^{\dot\gamma}_l[\bar\omega^i_{\dot\alpha}, \bar{D}^j_{\dot\beta}\hat{\bar\omega}^{l\,1}_{\dot\gamma}] = K^{ij}_{\dot\alpha\dot\beta}\,, \tag{4.33}$$

where we have abbreviated

$$K^{ij}_{\dot\alpha\dot\beta} \equiv \frac{1}{2}\epsilon_{\dot\alpha\dot\beta}\epsilon^{ijkl}\hat{W}^1_{kl} + \frac{1}{4}C^{m\delta,n\epsilon}\{\partial_{m\delta}\bar\omega^i_{\dot\alpha}, \partial_{n\epsilon}\bar\omega^j_{\dot\beta}\} + \{\bar\omega^i_{\dot\alpha}, \bar{D}^j_{\dot\beta}(\theta\hat\omega^1)\}\,. \tag{4.34}$$

In order to simplify notation, let us introduce the shorthand index notation $\bar{A} = \left({\dot\alpha \atop i}\right)$, etc., and rewrite (4.33) as

$$\bar{D}_{\bar A}\hat{\bar\omega}^1_{\bar B} - \bar\theta^{\bar C}[\bar\omega_{\bar A}, \bar{D}_{\bar B}\hat{\bar\omega}^1_{\bar C}] = K_{\bar A\bar B}\,. \tag{4.35}$$

This equation might be iterated to give the solution for $\bar{D}_{\bar A}\hat{\bar\omega}^1_{\bar B}$,

$$\bar{D}_{\bar A}\hat{\bar\omega}^1_{\bar B} = \sum_{|\bar I|\leq 8}(-)^{\lfloor\frac{|\bar I|}{2}\rfloor}\,\bar\theta^{\bar I}\,[\bar\omega, K\}_{\bar I,\bar A\bar B}\,, \tag{4.36}$$

where "$\lfloor\ \rfloor$" denotes the Gauß bracket and

$$[\bar\omega, K\}_{\bar I,\bar A\bar B} \equiv [\bar\omega_{\bar A}, [\bar\omega_{\bar B}, [\bar\omega_{\bar A_1}, \cdots [\bar\omega_{\bar A_{|\bar I|-2}}, K_{\bar A_{|\bar I|-1}\bar A_{|\bar I|}}\}\cdots\}\}\}\,. \tag{4.37}$$

Note that the sum in (4.36) is finite which is due to the nilpotency of the $\bar\theta$ in front of the bracket in (4.35). Therefore the first order contribution of the gauge potential $\hat{\bar\omega}^i_{\dot\alpha}$ is given by

$$\hat{\bar\omega}^1_{\bar A} = \bar{D}_{\bar A}(\theta\hat\omega^1) - \bar\theta^{\bar B}\sum_{|\bar I|\leq 8}(-)^{\lfloor\frac{|\bar I|}{2}\rfloor}\,\bar\theta^{\bar I}\,[\bar\omega, K\}_{\bar I,\bar A\bar B}\,. \tag{4.38}$$

[7] See also appendix A.

Now we are able to write down the field expansions for the remaining fields. The gauge potential $\hat{A}^1_{\alpha\dot\beta}$ reads as

$$\hat{A}^1_{\alpha\dot\beta} = \frac{1}{8}(\nabla_{i\alpha}\hat{\bar\omega}^{i\,1}_{\dot\beta} - \bar\nabla^i_{\dot\beta}\hat\omega^1_{i\alpha} + \frac{1}{2}C^{m\delta,n\epsilon}\{\partial_{m\delta}\omega_{i\alpha}, \partial_{n\epsilon}\bar\omega^i_{\dot\beta}\})\,, \tag{4.39}$$

which follows directly from (4.1c). The definitions (4.5) finally give us

$$\hat{\bar\chi}^{\,1}_{i\dot\beta} = -\frac{1}{2}\epsilon^{\alpha\beta}\left(\nabla_{i\alpha}\hat{A}^1_{\beta\dot\beta} - \nabla_{\beta\dot\beta}\hat\omega^1_{i\alpha} + \frac{1}{2}C^{m\delta,n\epsilon}\{\partial_{m\delta}\omega_{i\alpha}, \partial_{n\epsilon}A_{\beta\dot\beta}\}\right), \tag{4.40a}$$

$$\hat\chi^{i\,1}_{\beta} = -\frac{1}{2}\epsilon^{\dot\alpha\dot\beta}\left(\bar\nabla^i_{\dot\alpha}\hat{A}^1_{\beta\dot\beta} + \nabla_{\beta\dot\beta}\hat{\bar\omega}^{i\,1}_{\dot\alpha} - \frac{1}{2}C^{m\delta,n\epsilon}\{\partial_{m\delta}\bar\omega^i_{\dot\alpha}, \partial_{n\epsilon}A_{\beta\dot\beta}\}\right). \tag{4.40b}$$

Thus, we have computed the contributions to first order in the deformation. Now one could proceed further and compute the higher order contributions.

Finally, the field expansions of the undeformed superfields, which are given in the appendix A, have to be substituted into (4.40). But this is not too illuminating and we therefore refrain from doing this. We rather concentrate ourselves on the zeroth order components, i.e., those which do not contain the odd coordinates, since these will eventually give us the deformed field equations.

4.3 Field equations

The next thing we need to compute is the zeroth order components of the superfield equations (4.11) and (4.12). However, before we can derive them we have to discuss some preliminaries. In the equations (4.11) and (4.12) products of the form $\theta^I \star \theta^J$ do appear. Therefore we need to know their explicit zeroth order form. The first step in this direction is to show that

$$\begin{aligned} \theta^{A_1}\star\cdots\star\theta^{A_n} &= \theta^{A_1}\cdots\theta^{A_n} + \sum \text{all possible contractions} \\ &= \theta^{A_1}\cdots\theta^{A_n} + \sum_{i<j}\theta^{A_1}\cdots\,\overline{\theta^{A_i}\cdots\theta^{A_j}}\cdots\theta^{A_n} + \cdots\,, \end{aligned} \tag{4.41}$$

with the indices $A_k = (i_k\alpha_k)$ and

$$\overline{\theta^{A_i}\theta^{A_j}} \equiv \frac{\hbar}{2}C^{A_iA_j}\,. \tag{4.42}$$

Equation (4.41) resembles the fermionic Wick theorem. The signs have to be taken as in the fermionic Wick theorem, i.e.,

$$\overline{\theta^{A_i}\theta^{A_j}\theta^{A_k}} = -\frac{\hbar}{2}C^{A_iA_k}\theta^{A_j}\,,$$

for instance. The proof of (4.41) can easily be done by induction. For $n = 2$, equation (4.41) is obviously satisfied. For $n > 2$, one first shows that

$$(\theta^{A_1}\cdots\theta^{A_n})\star\theta^{A_{n+1}} = \theta^{A_1}\cdots\theta^{A_n}\theta^{A_{n+1}} + \sum_{i=1}^{n}\theta^{A_1}\cdots\,\overline{\theta^{A_i}\cdots\theta^{A_{n+1}}}\,, \tag{4.43}$$

from which then the assertion is immediate.

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Remember that any $\hat{f} \in \mathbb{R}_\hbar^{(4|16)}$ can be expanded in terms of the odd coordinates. In order to simplify notation, let us define a projector $\pi_\circ$ projecting onto the zeroth order component, i.e.,

$$\pi_\circ : \hat{f}(y, \theta, \bar{\theta}) \mapsto \mathring{\hat{f}}(y)\,. \tag{4.44}$$

Then we have

$$\begin{aligned} \pi_\circ(\theta^I \star \theta^J) &= \pi_\circ((\theta^{A_1} \cdots \theta^{A_n}) \star (\theta^{B_1} \cdots \theta^{B_m})) \\ &= \delta_{nm} \frac{(-)^{\frac{n}{2}(n-1)} \hbar^n}{2^n\, n!} \sum_{\{i,j\}} \epsilon_{i_1 \cdots i_n} \epsilon_{j_1 \cdots j_n} C^{A_{i_1} B_{j_1}} \cdots C^{A_{i_n} B_{j_n}}\,. \end{aligned} \tag{4.45}$$

The proof of (4.45) is rather obvious. Starting point is the equation (4.43). It follows from this equation that at zeroth order only such terms of the product $\theta^I \star \theta^J$ contribute for which $|I| = |J|$, because if $|I| \neq |J|$ equation (4.43) shows that there will be no fully contracted term and hence no contribution at zeroth order. Then symmetry arguments, (4.41) and (4.43) lead to

$$\pi_\circ((\theta^{A_1} \cdots \theta^{A_n}) \star (\theta^{B_1} \cdots \theta^{B_n})) = \pi_\circ(\theta^{A_1} \star \cdots \star \theta^{A_n} \star \theta^{B_1} \star \cdots \star \theta^{B_n})\big|_{\text{no } \overline{A_i A_j},\, \overline{B_i B_j}}\,,$$

i.e., the zeroth order component consists of all possible contractions but without contractions among the θ^{A_i}s (θ^{B_i}s). Since from the very beginning the θ^{A_i}s (θ^{B_i}s) appear totally antisymmetrized, we obtain the Levi-Civita symbols on the right hand side of (4.45). The factor of $1/n!$ provides the correct number of terms appearing after summation and the 2^n comes from the n contractions. The sign $(-)^{\frac{n}{2}(n-1)}$ needs to be included, since in order to get the term proportional to $\epsilon_{i_1 \cdots i_n} \epsilon_{j_1 \cdots j_n} = \epsilon_{1 \cdots n} \epsilon_{1 \cdots n} = 1$ one has to anticommute $\frac{n}{2}(n-1)$ thetas. As a corallary of (4.45) it follows that

$$\pi_\circ(\theta^I \star \theta^J) = \pi_\circ(\theta^J \star \theta^I)\,. \tag{4.46}$$

Let $\hat{f}$ and $\hat{g}$ be $\mathfrak{g}$-valued elements of $\mathbb{R}_\hbar^{(4|16)}$. In the equations (4.11) and (4.12) always commutators or anticommutators of superfields appear. Therefore we are interested in $\pi_\circ([\hat{f}, \hat{g}\}_\star)$. To compute this expression, we expand $\hat{f}$ and $\hat{g}$ as

$$\hat{f} = \mathring{\hat{f}}(y) + \sum_I \hat{f}_I(y) \theta^I + \cdots \qquad \text{and} \qquad \hat{g} = \mathring{\hat{g}}(y) + \sum_J \hat{g}_J(y) \theta^J + \cdots\,, \tag{4.47}$$

where the dots represent terms containing at least one $\bar{\theta}$. Then we have to distinguish three cases, namely $(p_{\hat{f}}, p_{\hat{g}}) = (0,0)$, $(p_{\hat{f}}, p_{\hat{g}}) = (1,1)$ and $(p_{\hat{f}}, p_{\hat{g}}) = (0,1)$ leading to

$$\pi_\circ([\hat{f}, \hat{g}]_\star) = [\mathring{\hat{f}}, \mathring{\hat{g}}] + \sum_{|I|=|J|} (-)^{p_I} [\hat{f}_I, \hat{g}_J]\ \pi_\circ(\theta^I \star \theta^J)\,, \tag{4.48a}$$

$$\pi_\circ(\{\hat{f}, \hat{g}\}_\star) = \{\mathring{\hat{f}}, \mathring{\hat{g}}\} + \sum_{|I|=|J|} \{\hat{f}_I, \hat{g}_J\}\ \pi_\circ(\theta^I \star \theta^J)\,, \tag{4.48b}$$

$$\pi_\circ([\hat{f}, \hat{g}]_\star) = [\mathring{\hat{f}}, \mathring{\hat{g}}] + \sum_{|I|=|J|} (\hat{f}_I \hat{g}_J - (-)^{p_I} \hat{g}_J \hat{f}_I)\ \pi_\circ(\theta^I \star \theta^J)\,, \tag{4.48c}$$

respectively. In deriving these expressions we have used (4.46).

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We have now all ingredients to give the zeroth order components of (4.11) and (4.12). For brevity, let us define $T^{IJ} \equiv \pi_\circ(\theta^I \star \theta^J)$. Remember that T^{IJ} is symmetric, i.e., $T^{IJ} = T^{JI}$. Putting everything together, the equations (4.11) become

$$
\begin{aligned}
\epsilon^{\alpha\beta}\mathring{\hat{\nabla}}_{\alpha\dot\alpha}\mathring{\hat{\chi}}^i_\beta + \frac{1}{2}\epsilon^{ijkl}[\mathring{\hat{W}}_{kl}, \mathring{\bar{\hat{\chi}}}_{j\dot\alpha}] &= -\epsilon^{\alpha\beta}\sum_{|I|=|J|}(\hat{A}_{\alpha\dot\alpha|I}\hat{\chi}^i_{\beta|J} - (-)^{p_I}\hat{\chi}^i_{\beta|J}\hat{A}_{\alpha\dot\alpha|I})T^{IJ} - \\
&\quad -\frac{1}{2}\epsilon^{ijkl}\sum_{|I|=|J|}(\hat{W}_{kl|I}\bar{\hat{\chi}}_{j\dot\alpha|J} - (-)^{p_I}\bar{\hat{\chi}}_{j\dot\alpha|J}\hat{W}_{kl|I})T^{IJ}\,, \qquad (4.49a) \\
\epsilon^{\dot\alpha\dot\beta}\mathring{\hat{\nabla}}_{\alpha\dot\alpha}\mathring{\bar{\hat{\chi}}}_{i\dot\beta} + [\mathring{\hat{W}}_{ij}, \mathring{\hat{\chi}}^j_\alpha] &= -\epsilon^{\dot\alpha\dot\beta}\sum_{|I|=|J|}(\hat{A}_{\alpha\dot\alpha|I}\bar{\hat{\chi}}_{i\beta|J} - (-)^{p_I}\bar{\hat{\chi}}_{i\beta|J}\hat{A}_{\alpha\dot\alpha|I})T^{IJ} - \\
&\quad - \sum_{|I|=|J|}(\hat{W}_{ij|I}\hat{\chi}^j_{\alpha|J} - (-)^{p_I}\hat{\chi}^j_{\alpha|J}\hat{W}_{ij|I})T^{IJ}\,, \qquad (4.49b)
\end{aligned}
$$

while the equations of motion for the gauge field and the scalar multiplet are

$$
\begin{aligned}
&\epsilon^{\dot\alpha\dot\beta}\mathring{\hat{\nabla}}_{\gamma\dot\alpha}\mathring{\bar{\hat{f}}}_{\dot\beta\dot\gamma} + \epsilon^{\alpha\beta}\mathring{\hat{\nabla}}_{\alpha\dot\gamma}\mathring{\hat{f}}_{\beta\gamma} - \frac{1}{4}\epsilon^{ijkl}[\mathring{\hat{\nabla}}_{\gamma\dot\gamma}\mathring{\hat{W}}_{ij}, \mathring{\hat{W}}_{kl}] - \{\mathring{\hat{\chi}}^i_\gamma, \mathring{\bar{\hat{\chi}}}_{i\dot\gamma}\} = \\
&\quad = - \sum_{|I|=|J|}(-)^{p_I}\left\{\epsilon^{\dot\alpha\dot\beta}[\hat{A}_{\gamma\dot\alpha|I}, \bar{\hat{f}}_{\dot\beta\dot\gamma|J}] + \epsilon^{\alpha\beta}[\hat{A}_{\alpha\dot\gamma|I}, \hat{f}_{\beta\gamma|J}]\right\}T^{IJ} + \\
&\qquad + \sum_{|I|=|J|}\left\{(-)^{p_I}\frac{1}{4}\epsilon^{ijkl}[(\hat{\nabla}_{\gamma\dot\gamma}\hat{W}_{ij})_I, \hat{W}_{kl|J}] + \{\hat{\chi}^i_{\gamma|I}, \bar{\hat{\chi}}_{i\dot\gamma|J}\}\right\}T^{IJ}\,, \qquad (4.50a) \\
&\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}\mathring{\hat{\nabla}}_{\alpha\dot\alpha}\mathring{\hat{\nabla}}_{\beta\dot\beta}\mathring{\hat{W}}_{ij} - \frac{1}{4}\epsilon^{klmn}[\mathring{\hat{W}}_{mn}, [\mathring{\hat{W}}_{kl}, \mathring{\hat{W}}_{ij}]] = \\
&\quad = \frac{1}{2}\epsilon_{ijkl}\epsilon^{\alpha\beta}\{\mathring{\hat{\chi}}^k_\alpha, \mathring{\hat{\chi}}^l_\beta\} + \epsilon^{\dot\alpha\dot\beta}\{\mathring{\bar{\hat{\chi}}}_{i\dot\alpha}, \mathring{\bar{\hat{\chi}}}_{j\dot\beta}\} - \\
&\qquad - \sum_{|I|=|J|}(-)^{p_I}\left\{\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}[\hat{A}_{\alpha\dot\alpha|I}, (\hat{\nabla}_{\beta\dot\beta}\hat{W}_{ij})_J] - \frac{1}{4}\epsilon^{klmn}[\hat{W}_{mn|I}, [\hat{W}_{kl}, \hat{W}_{ij}]_J]\right\}T^{IJ} + \\
&\qquad + \sum_{|I|=|J|}\left\{\frac{1}{2}\epsilon_{ijkl}\epsilon^{\alpha\beta}\{\hat{\chi}^k_{\alpha|I}, \hat{\chi}^l_{\beta|J}\} + \epsilon^{\dot\alpha\dot\beta}\{\bar{\hat{\chi}}_{i\dot\alpha|I}, \bar{\hat{\chi}}_{j\dot\beta|J}\}\right\}T^{IJ}\,. \qquad (4.50b)
\end{aligned}
$$

These equations and the field expansions (4.40) together with the undeformed superfield expansions given in the appendix A allow us to write down the deformed field equations. The derivations of the zeroth order components of (4.40) is pretty lengthy but straightforward. We therefore will not present them here, but only give the results. We ultimately find

$$
\begin{aligned}
\mathring{\hat{W}}_{ij} &= \mathring{W}_{ij} + \frac{\hbar}{2}C^{m\delta,n\epsilon}\epsilon_{\delta\epsilon}\{\mathring{W}_{mi}, \mathring{W}_{jn}\} + \mathcal{O}(\hbar^2)\,, \qquad (4.51a) \\
\mathring{\hat{A}}_{\alpha\dot\beta} &= \mathring{A}_{\alpha\dot\beta} + \frac{\hbar}{4}C^{m\delta,n\epsilon}\epsilon_{\alpha\delta}\{\mathring{W}_{mn}, \mathring{A}_{\epsilon\dot\beta}\} + \mathcal{O}(\hbar^2)\,, \qquad (4.51b) \\
\mathring{\bar{\hat{\chi}}}_{i\dot\beta} &= \mathring{\bar{\chi}}_{i\dot\beta} + \frac{\hbar}{96}C^{m\delta,n\epsilon}\Big[11\epsilon_{\delta\epsilon}\Big(\{\mathring{W}_{mn}, \mathring{\bar{\chi}}_{i\dot\beta}\} - 2\{\mathring{W}_{in}, \mathring{\bar{\chi}}_{m\dot\beta}\}\Big) - \\
&\quad - 5\Big(\epsilon_{mnij}\{\mathring{A}_{\delta\dot\beta}, \mathring{\chi}^j_\epsilon\}\Big)\Big] + \mathcal{O}(\hbar^2)\,, \qquad (4.51c)
\end{aligned}
$$

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$$
\begin{aligned}
\mathring{\hat{\chi}}^i_\beta = \mathring{\chi}^i_\beta + \frac{\hbar}{16} C^{m\delta,n\epsilon} \Bigg[& \left\{ \mathring{W}_{mn}, \frac{4}{3}\epsilon_{\epsilon\delta}\mathring{\chi}^i_\beta - \frac{11}{3}\epsilon_{\delta\beta}\mathring{\chi}^i_\epsilon \right\} - \\
& - \delta^i_m \left\{ \mathring{W}_{ln}, \frac{4}{3}\epsilon_{\epsilon\delta}\mathring{\chi}^l_\beta + \frac{7}{3}\epsilon_{\epsilon\beta}\mathring{\chi}^l_\delta - \frac{2}{3}\epsilon_{\delta\beta}\mathring{\chi}^l_\epsilon \right\} - \\
& - \epsilon_{\beta\epsilon}\epsilon^{\dot\alpha\dot\beta} \left\{ \mathring{A}_{\delta\dot\alpha}, 12\delta^i_m \mathring{\bar{\chi}}_{n\dot\beta} - \frac{1}{2}\delta^i_n \mathring{\bar{\chi}}_{m\dot\beta} \right\} \Bigg] + \mathcal{O}(\hbar^2)\,. \qquad (4.51\text{d})
\end{aligned}
$$

Now one substitutes these expressions into the equations (4.49) and (4.50) and uses the undeformed expansions given in the appendix A to obtain the deformed super Yang-Mills equations to first order in $\hbar$, e.g., $\epsilon^{\alpha\beta}\mathring{\nabla}_{\alpha\dot\alpha}\mathring{\chi}^i_\beta + \frac{1}{2}\epsilon^{ijkl}[\mathring{W}_{kl}, \mathring{\bar{\chi}}_{j\dot\alpha}] = \mathcal{O}(\hbar)$ (note that solving this equation consistently together with the other equations of motion makes obviously the fields on the left-hand side $\hbar$-dependent). Actually performing this task leads to both unenlightening and complicated looking expressions, so we refrain from writing them down. To proceed in a realistic manner, one can constrain the deformation parameters to obtain manageable equations of motion.

For instance, in order to compare the deformed equations of motion with Seiberg's deformed $\mathcal{N} = 1$ equations,[8] one would have to restrict the deformation matrix $C^{i\alpha,j\beta}$ properly and to put some of the fields, e.g., W_{ij}, to zero. Additionally, one would have to rotate the fermion field content such that the symplectic reality condition induced by (2.6) holds. Here, however, one encounters the subtlety that on $\mathcal{N} = 1$ superspace with euclidean signature the gauge potentials are necessarily complex (cf., e.g., reference [35]). For these and other reasons, this comparison would carry us too far afield from the main thread of development of the present paper. Therefore, we shall discuss this issue in our forthcoming work [36].

5. Conclusions

In this paper we have proposed a way of deforming $\mathcal{N} = 4$ super Yang-Mills theory. The starting point was the constraint equations on the deformed superspace $\mathbb{R}^{(4|16)}_\hbar$ from which we derived the deformed superspace equations of motion. By using a generalization of the Seiberg-Witten map to superspace, we gave a systematic procedure of constructing the deformed superfields order by order in the deformation $\hbar$. Eventually, these yield deformed equations of motion on $\mathbb{R}^4$ with a larger number of deformation parameters than in the case of $\mathcal{N} = 1, 2$ deformations.

Generalizing the string theory side of the derivation of Seiberg-Witten maps seems to be nontrivial. The graviphoton used to deform the fermionic coordinates belongs to the R-R sector, while the gauge field strength causing the deformation in the bosonic case sits in the NS-NS sector. This implies, that the field strengths appear on different footing in the vertex operators of the appropriate string theory (type II with $\mathcal{N} = 2$, $d = 4$). A first step might be to consider a "pure gauge" configuration in which the gluino and gluon field strengths vanish. The corresponding vertex operator in Berkovits' hybrid formalism on the

[8] Or similarly in the case of the deformed $\mathcal{N} = 2$ equations in $\mathcal{N} = 1$ superspace language [34, 12].

boundary of the worldsheet of an open string contains the terms [37],

$$V = \frac{1}{2\alpha'} \int d\tau \left(\dot{\theta}^\alpha \omega_\alpha + \dot{X}^\mu A_\mu - i\sigma^\mu_{\alpha\dot{\alpha}} \dot{\theta}^\alpha \bar{\theta}^{\dot{\alpha}} A_\mu \right),$$

with the formal (classical) gauge transformations $\delta_\lambda \omega_\alpha = D_\alpha \lambda$ and $\delta_\lambda A_\mu = \partial_\mu \lambda$. From here, one may proceed exactly as in [28] using the deformation of [12]: regularization of the action by Pauli-Villars[9] and point-splitting procedures lead to an undeformed and a deformed gauge invariance, respectively. Although on flat euclidean space, pure gauge is trivial, the two different gauge transformations obtained are not.

More general, a Seiberg-Witten map is a translation rule between two physically equivalent field theories. The fact that our choice of the deformation (3.13) generically breaks half of the supersymmetry is not in contradiction with the existence of a Seiberg-Witten map, but may be seen analogously to the purely bosonic case: in both the commutative and noncommutative theories, particle Lorentz invariance is broken which is due to the background field (B-field).

Furthermore, there are several open questions which should be clarified. We only indicated the construction of the deformed superfields, since already to first order in $\hbar$ the computations became pretty lengthy. Therefore the question is whether the changes in the deformed superfields in comparison to the undeformed ones are polynomial in $\hbar$ as suggested by the nilpotency of the star product. On the other hand, it remains to clarify whether the one-to-one correspondence between the deformed equations of motion and the constraint equations is still valid on $\mathbb{R}^{(4|16)}_\hbar$.

Another point concerns the Seiberg-Witten map proposed above: it might be used to shed light on the question of supersymmetry breaking in different approaches to the deformation of the fermionic coordinates. E.g., in [12], the approach also used above, the deformation breaks supersymmetry in general, while [2] uses a supersymmetric deformation preserving the algebra at the price of losing chirality. In the latter case, one should also be able to construct a Seiberg-Witten map. However, here the $\bar{D}$s are no longer derivations with respect to the star product and hence they should be modified properly (by using Kontsevich's formality map) in order to follow the same steps as presented above. For this approach, the Seiberg-Witten map would relate two (fully) supersymmetric theories. A third point of view is found in [9]: a deformation of the fermionic coordinates is also induced by a self-dual graviphoton background but later on compensated by introducing a gluino background so that the ordinary superspace is restored.

Finally, it would be illuminating to explore the connection of the constraint equations and the auxiliary linear system of partial differential equations (to which the constraint equations are the compatibility condition) on the deformed superspace. In the undeformed case, the existence of such a linear system [39, 40, 41] leads to the integrability of $\mathcal{N} = 4$ super Yang-Mills theory. Therefore dressing [42] and splitting [43] methods can be applied [25, 44, 45, 46] for its solving. It would be interesting to generalize these methods not only to the noncommutative case as in [47]–[53], but also to the nonanticommutative deformation of $\mathcal{N} = 4$ super Yang-Mills theory.

[9] Pauli-Villars was applied to supergravity, e.g., in [38].

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A. Superfield expansions for vanishing deformation

In this appendix we shall review how one can construct the superfields from their leading components. Following [26, 27], we impose the gauge condition

$$\theta\omega - \bar{\theta}\bar{\omega} = \theta^{i\alpha}\omega_{i\alpha} + \bar{\theta}_i^{\dot{\alpha}}\bar{\omega}^i_{\dot{\alpha}} = 0 \tag{A.1}$$

in order to remove the superfluous gauge degrees of freedom associated with the $\theta^{i\alpha}$ and $\bar{\theta}_i^{\dot{\alpha}}$ coordinates. Moreover, we need some recursion operator, $\mathcal{D}$, which leads to the proper field expansions. We take the following form [26, 27]

$$\begin{aligned}\mathcal{D}f &\equiv (\theta D + \bar{\theta}\bar{D})f \\ &= (\theta^{i\alpha}D_{i\alpha} + \bar{\theta}_{i\dot{\alpha}}\bar{D}^{i\dot{\alpha}})f = (\theta^{i\alpha}D_{i\alpha} - \bar{\theta}_i^{\dot{\alpha}}\bar{D}^i_{\dot{\alpha}})f\,,\end{aligned} \tag{A.2}$$

where $f \in \mathbb{R}^{(4|16)}$. It then follows immediately that in the gauge (A.1) the recursion operator $\mathcal{D}$ is the same as the covariant one, i.e.,

$$\mathcal{D} = \theta\nabla + \bar{\theta}\bar{\nabla}\,. \tag{A.3}$$

By using the undeformed version of the constraint equations (4.1), we obtain after some simple algebraic manipulations

$$(1+\mathcal{D})\omega_{i\alpha} = 2\bar{\theta}_i^{\dot{\alpha}}A_{\alpha\dot{\alpha}} - 2\epsilon_{\alpha\beta}\theta^{j\beta}W_{ij}\,, \tag{A.4a}$$

$$(1+\mathcal{D})\bar{\omega}^i_{\dot{\alpha}} = 2\theta^{i\alpha}A_{\alpha\dot{\alpha}} - \epsilon_{\dot{\alpha}\dot{\beta}}\epsilon^{ijkl}\bar{\theta}_j^{\dot{\beta}}W_{kl}\,. \tag{A.4b}$$

Finally, the (undeformed) equations (4.5), (4.7) and (4.8) give us the remaining relations

$$\mathcal{D}A_{\alpha\dot{\alpha}} = -\epsilon_{\alpha\beta}\theta^{i\beta}\bar{\chi}_{i\dot{\alpha}} + \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}_i^{\dot{\beta}}\chi^i_\alpha\,, \tag{A.4c}$$

$$\mathcal{D}W_{ij} = \epsilon_{ijkl}\theta^{k\alpha}\chi^l_\alpha - \bar{\theta}_i^{\dot{\alpha}}\bar{\chi}_{j\dot{\alpha}} + \bar{\theta}_j^{\dot{\alpha}}\bar{\chi}_{i\dot{\alpha}}\,, \tag{A.4d}$$

$$\mathcal{D}\chi^i_\alpha = -2\theta^{i\beta}f_{\alpha\beta} + \frac{1}{2}\epsilon_{\alpha\beta}\epsilon^{iklm}\theta^{j\beta}[W_{lm}, W_{jk}] - \epsilon^{ijkl}\bar{\theta}_j^{\dot{\alpha}}\nabla_{\alpha\dot{\alpha}}W_{kl}\,, \tag{A.4e}$$

$$\mathcal{D}\bar{\chi}_{i\dot{\alpha}} = 2\theta^{j\alpha}\nabla_{\alpha\dot{\alpha}}W_{ij} + 2\bar{\theta}_i^{\dot{\beta}}f_{\dot{\alpha}\dot{\beta}} + \frac{1}{2}\epsilon_{\dot{\alpha}\dot{\beta}}\epsilon^{jklm}\bar{\theta}_j^{\dot{\beta}}[W_{lm}, W_{ik}]\,, \tag{A.4f}$$

as one may readily verify. The equations (A.4) are regarded as a recursive definition of the superfields, i.e., by iterating these equations we can reconstruct the superfields order by order in the odd coordinates from their leading components. To exemplify our situation,

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let us write down the expansions of the superfields $A_{\alpha\dot\alpha}$, W_{ij}, χ^i_α and $\bar\chi_{i\dot\alpha}$ up to quadratic order in θ (no $\bar\theta$s),

$$
\begin{aligned}
A_{\alpha\dot\alpha} &= \mathring{A}_{\alpha\dot\alpha} + \epsilon_{\alpha\beta}\mathring{\bar\chi}_{i\dot\alpha}\theta^{i\beta} - \epsilon_{\alpha\beta}\mathring{\nabla}_{\alpha\dot\alpha}\mathring{W}_{ij}\theta^{i\beta}\theta^{j\gamma} + \cdots\,, && \text{(A.5a)}\\
W_{ij} &= \mathring{W}_{ij} - \epsilon_{ijkl}\mathring{\chi}^l_\alpha\theta^{k\alpha} - \epsilon_{ijkl}\left(\delta^l_m\mathring{f}_{\beta\alpha} + \frac{1}{4}\epsilon_{\beta\alpha}\epsilon^{lnij}[\mathring{W}_{ij},\mathring{W}_{mn}]\right)\theta^{k\alpha}\theta^{m\beta} + \cdots\,, && \text{(A.5b)}\\
\chi^i_\alpha &= \mathring{\chi}^i_\alpha - \left(2\delta^i_j\mathring{f}_{\beta\alpha} + \frac{1}{2}\epsilon_{\beta\alpha}\epsilon^{iklm}[\mathring{W}_{lm},\mathring{W}_{jk}]\right)\theta^{j\beta} + \\
&\quad + \left\{\frac{1}{2}\delta^i_j\epsilon^{\dot\alpha\dot\beta}\left(\epsilon_{\gamma\alpha}\mathring{\nabla}_{\beta\dot\alpha}\mathring{\bar\chi}_{k\dot\beta} + \epsilon_{\gamma\beta}\mathring{\nabla}_{\alpha\dot\alpha}\mathring{\bar\chi}_{k\dot\beta}\right) - \right.\\
&\qquad \left. - \frac{1}{4}\epsilon_{\alpha\beta}\epsilon^{ipmn}\left(\epsilon_{jkpq}[\mathring{W}_{mn},\mathring{\chi}^q_\gamma] + \epsilon_{mnkq}[\mathring{W}_{jp},\mathring{\chi}^q_\gamma]\right)\right\}\theta^{j\beta}\theta^{k\gamma} + \cdots\,, && \text{(A.5c)}\\
\bar\chi_{i\dot\alpha} &= \mathring{\bar\chi}_{i\dot\alpha} + 2\mathring{\nabla}_{\alpha\dot\alpha}\mathring{W}_{ij}\theta^{j\alpha} + \left(\epsilon_{ijkl}\mathring{\nabla}_{\alpha\dot\alpha}\mathring{\chi}^l_\beta + \epsilon_{\alpha\beta}[\mathring{W}_{ij},\mathring{\bar\chi}_{k\dot\alpha}]\right)\theta^{j\alpha}\theta^{k\beta} + \cdots\,, && \text{(A.5d)}
\end{aligned}
$$

where

$$
\mathring{f}_{\alpha\beta} = -\frac{1}{2}\epsilon^{\dot\alpha\dot\beta}[\mathring{\nabla}_{\alpha\dot\alpha},\mathring{\nabla}_{\beta\dot\beta}] = -\frac{1}{2}\epsilon^{\dot\alpha\dot\beta}\left(\partial_{\alpha\dot\alpha}\mathring{A}_{\beta\dot\beta} - \partial_{\beta\dot\beta}\mathring{A}_{\alpha\dot\alpha} + [\mathring{A}_{\alpha\dot\alpha},\mathring{A}_{\beta\dot\beta}]\right)\,.
$$

To arrive at (A.5) we have used the formal field expansion (3.5) and the equations (A.4c)–(A.4f). It is important to stress that the recursions do not involve the field equations. Therefore all superfields are well defined off-shell.

Now we are able to give the expansions of $\omega_{i\alpha}$ and $\bar\omega^i_{\dot\alpha}$. The equations (A.4a), (A.4b) then yield

$$
\begin{aligned}
\omega_{i\alpha} &= -\epsilon_{\alpha\beta}\mathring{W}_{ij}\theta^{j\beta} + \delta^j_i\mathring{A}_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}_j - \frac{2}{3}\epsilon_{\alpha\beta}\epsilon_{ijkl}\mathring{\chi}^l_\delta\theta^{j\beta}\theta^{k\delta} + \\
&\quad + \frac{2}{3}\epsilon_{\alpha\gamma}\left(2\delta^l_i\mathring{\bar\chi}_{k\dot\gamma} - \delta^l_k\mathring{\bar\chi}_{i\dot\gamma}\right)\theta^{k\gamma}\bar\theta^{\dot\gamma}_l + \frac{2}{3}\delta^j_i\epsilon_{\dot\alpha\dot\beta}\mathring{\chi}^k_\alpha\bar\theta^{\dot\alpha}_j\bar\theta^{\dot\beta}_k + \cdots\,, && \text{(A.5e)}\\
\bar\omega^i_{\dot\alpha} &= \delta^i_j\mathring{A}_{\alpha\dot\alpha}\theta^{j\alpha} - \frac{1}{2}\epsilon_{\dot\alpha\dot\beta}\epsilon^{ijkl}\mathring{W}_{kl}\bar\theta^{\dot\beta}_j - \frac{2}{3}\delta^i_j\epsilon_{\alpha\beta}\mathring{\bar\chi}_{k\dot\alpha}\theta^{j\alpha}\theta^{k\beta} + \\
&\quad + \frac{2}{3}\epsilon_{\dot\alpha\dot\gamma}\left(2\delta^i_k\mathring{\chi}^l_\gamma - \delta^l_k\mathring{\chi}^i_\gamma\right)\theta^{k\gamma}\bar\theta^{\dot\gamma}_l + \frac{2}{3}\epsilon_{\dot\alpha\dot\beta}\epsilon^{ijkl}\mathring{\bar\chi}_{l\dot\gamma}\bar\theta^{\dot\beta}_j\bar\theta^{\dot\gamma}_k + \cdots\,. && \text{(A.5f)}
\end{aligned}
$$

Here, we have also written down the $\bar\theta$, $\theta\bar\theta$ and $\bar\theta\bar\theta$ components as we need them in the discussion of the deformed superfields.

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